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A note on natural convection in a vertical slot

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The secondary and tertiary motions which were observed by Elder (1965) during his experimental study of natural convection in a vertical cavity have been obtained in a numerical solution of the Boussinesq equations. Aspect ratios up to 20, Rayleigh numbers up to 3000000, a Prandtl number of 1000, and up to almost 11000 effective mesh points were used.

An intriguing aspect of natural convection is the occurrence of secondary and tertiary motions in a vertical slot heated on one side and cooled on the other. Elder (1965) reported that, as the temperature difference between the two walls of a cavity containing a high Prandtl number fluid was increased, a steady secondary motion developed which was embedded in the single-roll, low temperature difference base flow. The secondary motion appeared as a vertical series of rolls of roughly rhombic cross-section. A further increase in the temperature difference resulted in the production of small tertiary rolls in the shear layers between the secondary rolls.

Elder demonstrated that the observed form of the motion could be constructed analytically by superimposing a spatially periodic disturbance on an approximate model of the base flow. Using marginal-stability analysis, criteria for the instability of the base flow and the development of the secondary motion have been calculated by Birikh (1966), Rudakov (1966), Vest & Arpaci (1969), Gill & Davey (1969), Birikh *et al.* (1969, 1972) and Gill & Kirkham (1970).

The physical interpretation of these analyses is that, for a cavity of given aspect ratio h containing a fluid of Prandtl number Pr ($= \nu/\kappa$, where ν and κ are the coefficients of kinematic viscosity and thermal diffusivity), there is a critical value Ra_c of the Rayleigh number Ra (defined here by $Ra = \Delta T \beta L^3 g/\nu\kappa$, where β is the coefficient of thermal expansion, g the gravitational acceleration, ΔT the temperature difference and L the distance between the walls of the cavity) at which the base flow becomes unstable to certain small disturbances. For $Ra > Ra_c$ these disturbances grow to form a spatially periodic motion which may be steady or unsteady, depending on whether the base flow is unstable to stationary or to travelling disturbances. The results of such an analysis are estimates of Ra_c and the wavelength of the disturbance which generates the secondary motion. The wavelength of the secondary motion which may exist for $Ra \gg Ra_c$ cannot, of course, be predicted.

The stability analyses that have been applied to this problem have been founded on the assumption that h is sufficiently large for the base flow and temperature field to be one-dimensional near the midheight of the cavity, where it is assumed that base-flow instability first occurs. The form of the vertical velocity profile across the cavity is deduced from the governing equations, which are reduced according to whether the base-flow heat transfer is conduction dominated, or alternatively, convection dominated to such an extent that distinct boundary layers are formed on each wall. Analysis of the instability of a conduction-regime base flow as performed by Birikh (1966), Rudakov (1966) and Birikh *et al.* (1972) is applicable only to the case of infinite h. In the case of a boundary-layer type of base flow (Vest & Arpaci 1969; Gill & Davey 1969; Birikh *et al.* 1969; Gill & Kirkham 1970) the effects of finite h can be partially incorporated in the expression for the base-flow vertical velocity profile and in the temperature gradient along the vertical centre-line. The latter is an additional input parameter and must be deduced independently.

Elder observed steady secondary motion in a cavity of aspect ratio of the order of 10–20† filled with a fluid with $Pr = 1000$ for $3\times10^5 \leqslant Ra \leqslant 3{\cdot}3\times10^6$. For these conditions the flow is well outside the conduction regime and is approaching the boundary-layer regime (see Gill & Davey 1969, p. 797).

The analysis of Gill & Kirkham (1970), which is an extension of that conducted by Gill & Davey to the case of high Pr, predicts that the flow is unstable to *travelling* disturbances at $Ra_c \sim 2{\cdot}5\times10^7$ for $h = 10$ and at $Ra_c \sim 3{\cdot}1\times10^6$ for $h = 20$; i.e. Elder's observations were made (subject to the uncertainty in his h) below their stability limit. Birikh *et al.* (1969) suggested that the initial instability (as Ra increases) is steady, but did not calculate Ra_c for the values of Pr of interest here.

Vest & Arpaci (1969) have calculated Ra_c for $Pr = 1000$: for $h = 10$, $Ra_c = 1{\cdot}75\times10^5$ and for $h = 20$, $Ra_c = 3\times10^5$. The latter estimate is in agreement with that made experimentally by Elder. However, their perturbation analysis omitted the term incorporating the vertical temperature gradient. This gradient can be expected to have a stabilizing effect (since the temperature increases with height) and their stability limit might be unrealistically low. Theirs is the only analysis to predict the wavenumber of the steady secondary flow; in view of their error, however, their prediction must be suspect. Their experiments confirm Elder's observation of steady secondary motion for Ra less than the critical value for travelling disturbances predicted by Gill & Kirkham (1970).

It is of considerable interest, therefore, to attempt to simulate convection for a finite aspect ratio and high Rayleigh number numerically to ascertain whether the fluid motion can be modelled with sufficient accuracy to generate the secondary and tertiary motions without making the assumptions characteristic of the stability analysis. Elder's (1966) computations did not reveal such motions, but he speculated that numerical oscillations in his finite-difference solutions were related to the possible existence of a periodic instability; de Vahl Davis & Thomas (1969) demonstrated that, in a vertical annulus, steady spatially periodic secondary flow can be induced by perturbing a steady single-roll

† The aspect ratio is not explicitly stated but is certainly not more than 19·6 (the maximum for his apparatus) nor less than 14·4 (the aspect ratio shown in his photographs).

solution. However, detailed solutions directly applicable to the cases studied experimentally by Elder (1965) have not appeared in the literature.

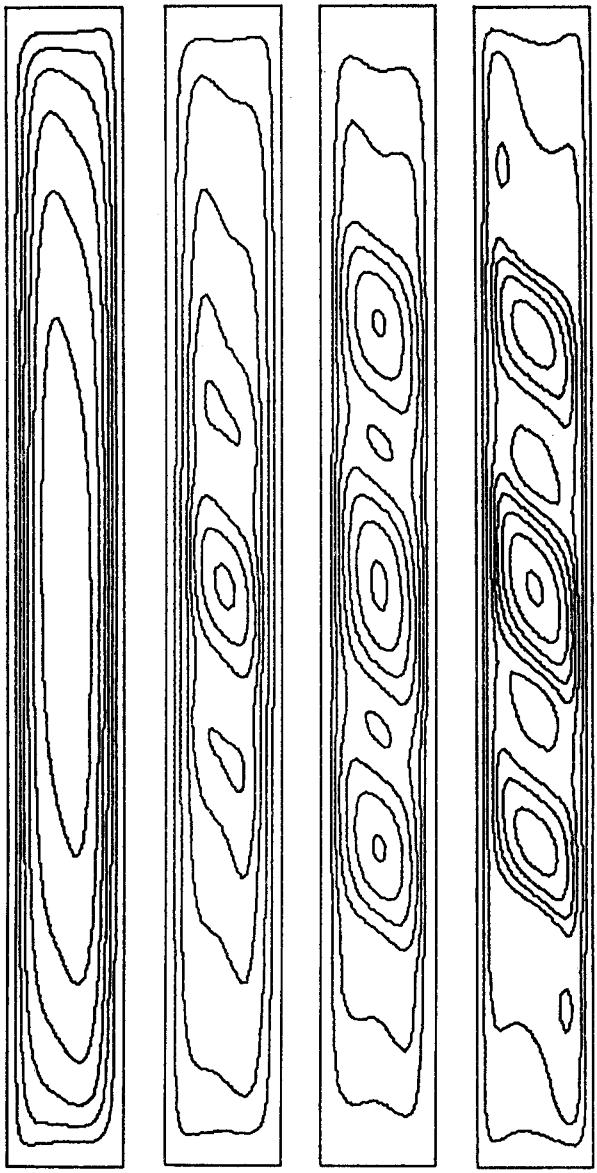
The numerical solutions shown in figures 1 and 2 were computed by solving conservative second-order finite-difference approximations to the steady-state Boussinesq equations using the method described by Mallinson & de Vahl Davis (1973). A 101×27 mesh was superimposed on the lower half of the cavity, the flow in which was assumed to be skew-symmetric about the horizontal centre-line. No deliberate attempt was made to perturb the solution fields; the random occurrence of round-off errors was evidently sufficient to initiate secondary motion for $Ra \gg Ra_c$.

The solution for $Ra = 2{\cdot}4 \times 10^5$ is a single roll. Elder estimated that secondary motion would appear at a critical Rayleigh number of 3×10^5. The numerical results show that, at $Ra = 5{\cdot}0 \times 10^5$, secondary motion is present near the centre of the cavity, suggesting that it is indeed instability in the centre of the flow field that leads to this secondary motion. As Ra is increased, the secondary motion strengthens and tertiary rolls develop as in the solution for $Ra = 9{\cdot}4 \times 10^5$ (corresponding to the value of Ra in figure 14 of Elder 1965). The tertiary motion is considerably stronger at $Ra = 3{\cdot}3 \times 10^6$. The wavelengths of the central rolls at $Ra = 9{\cdot}4 \times 10^5$ and $Ra = 3{\cdot}3 \times 10^6$ are 2·45 and 2·00 respectively, compared with Elder's measurements of 2·4 and 1·8. Since $Ra \gg Ra_c$, comparable theoretical estimates of the wavelengths are not available.

With the exception of the aspect ratio of the cavity ($h = 10$), the parameters for the solutions in figures 1(*c*) and (*d*) correspond to the conditions resulting in the flows photographed in a partially filled cavity of aspect ratio $\leqslant 20$. Using a 201×27 mesh over the lower half of the cavity, a solution was obtained at $Ra = 9{\cdot}4 \times 10^5$ and $h = 20$. The motion consisted of five secondary rolls (compared with three at $h = 10$) and four weak tertiary rolls, with an appearance generally similar to that shown in figure 1. However, the maximum value of the stream function increased by about one-third and the central wavelength decreased by about one-fifth as the aspect ratio was changed from 10 to 20. As an indication of the adequacy of the mesh at $h = 20$, it might be mentioned that the maximum value of the stream function changed by less than 1 % when the mesh was changed from 101×27 to 201×27.

A steady solution at $Ra = 3{\cdot}3 \times 10^6$ could only be obtained for the smaller aspect ratio, in confirmation of the theoretical analysis of Gill & Kirkham (1970), which predicts that for $Ra = 3{\cdot}3 \times 10^6$ and $h = 20$ the base flow is unstable to *travelling* disturbances. At the same Rayleigh number with $h = 10$ they predicted that the flow is stable to travelling disturbances.

Further comparison with experiment is afforded by the temperature distribution along the vertical centre-line. This was measured at $Ra = 3{\cdot}3 \times 10^6$ by Elder and the corresponding distribution determined numerically is illustrated in figure 3, which also shows the distributions along the vertical centre-line of the vorticity and stream function and the distribution along the heated wall of the Nusselt number. The temperature oscillates about the constant vertical temperature gradient, which is a characteristic feature of the boundary-layer base flow. The plateaux of constant temperature correspond to the secondary



(a) (b) (c) (d)

FIGURE 1. The flow field, as indicated by contours of the stream function ψ, in a cavity of aspect ratio 10 heated on one vertical wall and cooled on the other for four values of the Rayleigh number *Ra*. The hot wall is on the left side of each diagram and the Prandtl number of the fluid is 10^3. (*a*) $Ra = 2{\cdot}4 \times 10^5$. Contour levels for ψ are $-12{\cdot}5$ $(-12{\cdot}5)$ -75. (*b*) $Ra = 5{\cdot}0 \times 10^5$. Contour levels for ψ are -25, -50, -75 $(-12{\cdot}5)$ -125. (*c*) $Ra = 9{\cdot}4 \times 10^5$. Contour levels for ψ are -50 (-25) -200. (*d*) $Ra = 3{\cdot}3 \times 10^6$. Contour levels for ψ are -50, -100, -150 (-25) -275.

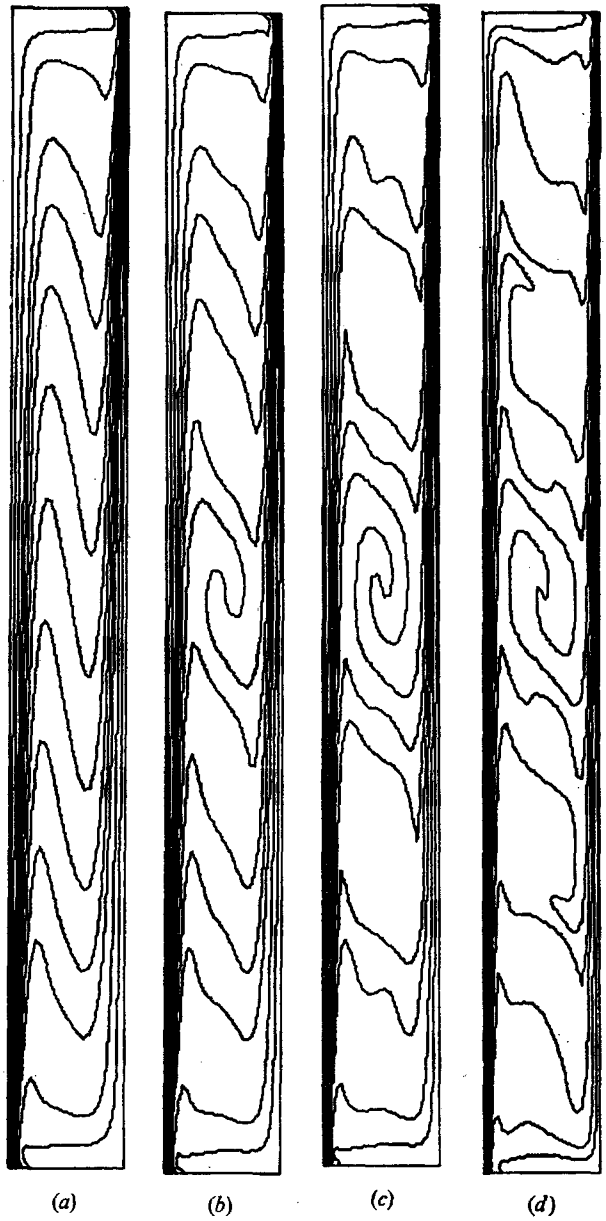


FIGURE 2. Isotherms corresponding to the solutions shown in figure 1. (*a*) $Ra = 2{\cdot}4 \times 10^5$. (*b*) $Ra = 5{\cdot}0 \times 10^5$. (*c*) $Ra = 9{\cdot}4 \times 10^5$. (*d*) $Ra = 3{\cdot}3 \times 10^6$. Contour levels in each case are $\pm 0{\cdot}4$, $\pm 0{\cdot}3$, $\pm 0{\cdot}2$, $\pm 0{\cdot}15$, $\pm 0{\cdot}1$, $\pm 0{\cdot}05$ and 0.

rolls, which have an essentially constant-temperature and constant-vorticity core. This is in agreement with the Batchelor (1956) model for flow involving closed streamlines when viscous diffusion of vorticity is negligible. In these rolls, convection is the dominant mode of heat transfer, as is reflected by the oscillations in the distribution of the local Nusselt number on the hot wall and

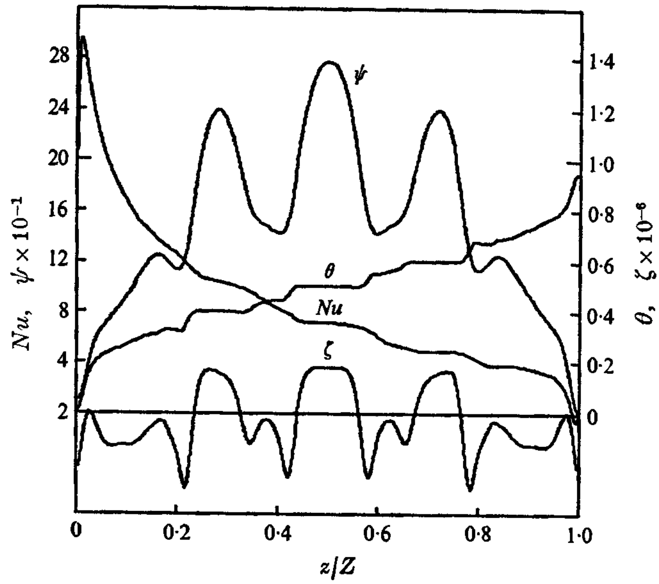


FIGURE 3. The distribution of the temperature θ, stream function ψ and vorticity ζ along the vertical centre-line and of the Nusselt number Nu on the hot wall, computed for $Ra = 3{\cdot}3\times10^6$, $Pr = 10^3$ and a cavity aspect ratio of 10. The co-ordinate z is measured from the top of the cavity and Z is the height of the cavity.

the way in which the isotherms (figure 2) closely follow the streamlines in the central region of the cavity.

Some differences between the results reported here and the experimental observations exist. For example, Elder observed that the wavelengths of the secondary rolls were not symmetrical about the horizontal centre-line and attributed this to the dependence of viscosity on temperature. In this respect the fact that the liquid in the experiments had a free upper surface, in contrast to the numerical model, which had a no-slip upper boundary condition, may also be significant.

The numerical model has, however, been successful in generating the complex secondary and tertiary motions originally observed by Elder, without making the assumptions regarding the form of the base flow and smallness of the disturbance amplitude that characterize and limit the range of validity of conventional stability analysis.

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